

A model for global compactness

Joint work with Inbar Oren and Assaf Rinot

This paper is dedicated with admiration to Menachem Magidor

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Definition (Keisler, Prikry)

Let U be an ultrafilter over an infinite cardinal κ , and $\theta < \kappa$ an infinite cardinal. U is *θ -indecomposable* iff for every $f: \kappa \rightarrow \theta$ there is $X \in U$ with $|f[X]| < \theta$. U is *uniform* iff every $X \in U$ has size κ .

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- For regular θ : U is θ -indecomposable iff U is closed under intersections of $\subseteq$ -decreasing sequences of length exactly θ .
- U is $[\chi, \nu)$ -indecomposable iff it is θ -indecomposable for every cardinal $\theta \in [\chi, \nu)$.
- U is ν -complete iff U is $[\omega, \nu)$ -indecomposable.

All of the following are consistency results, relative to large cardinals.

Year	Who	What
1983	Sheard	a non-measurable inaccessible κ carries a uniform UF that is θ -indec. for every uncountable $\theta < \kappa$
1986	Ben-David, Magidor	$\aleph_{\omega+1}$ carries a uniform UF that is θ -indec. for all uncountable $\theta < \aleph_\omega$; first model of $\square_{\aleph_\omega}^* + \neg\square_{\aleph_\omega}$
2000	Apter, Cummings	global version: for every singular λ , $\square_{\lambda, \text{cf}(\lambda)}$ holds and $\text{Refl}(S, < \text{cf}(\lambda))$ for some stationary $S \subseteq \lambda^+$
2015	Unger	$\aleph_{\omega_1+1}$ carries a uniform UF that is θ -indec. for all $\theta \in (\aleph_1, \aleph_{\omega_1})$
2025	Rinot, You, Yuan	a non-measurable <i>weakly compact</i> κ carries a uniform UF that is θ -indec. for every uncountable $\theta < \kappa$

Assume λ is singular and U is a uniform ultrafilter on λ^+ .

Kunen–Prikry (1971)

If U is θ -indecomposable for cofinally many regular $\theta < \lambda$, then U is *not* $\text{cf}(\lambda)$ -indecomposable.

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Hence, the possible final segment of indecomposability below λ^+ is $(\text{cf}(\lambda), \lambda)$.

The interval also requires $\neg \square_{\lambda, < \text{cf}(\lambda)}$

5 / 33

Prikry–Silver (1973)

For a uniform ultrafilter U on λ^+ , every stationary subset of $S_U := \{\alpha < \lambda^+ \mid U \text{ is } \text{cf}(\alpha)\text{-indecomposable}\}$ reflects at a point of cofinality $> \text{cf}(\lambda)$.

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Cummings–Foreman–Magidor (2001)

If $\square_{\lambda, < \text{cf}(\lambda)}$ holds, every stationary subset of λ^+ has a stationary subset that does not reflect at any point of cofinality $\geq \text{cf}(\lambda)$.

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Corollary

Let λ be singular and U a uniform $(\text{cf}(\lambda), \lambda)$ -indecomposable ultrafilter on λ^+ . Then $\square_{\lambda, < \text{cf}(\lambda)}$ fails.

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Corollary

Let λ be singular and U a uniform $(\text{cf}(\lambda), \lambda)$ -indecomposable ultrafilter on λ^+ . Then $\square_{\lambda, < \text{cf}(\lambda)}$ fails.

Proof. For regular $\theta \in (\text{cf}(\lambda), \lambda)$, $S_U \supseteq E_\theta^{\lambda^+}$ is stationary. If $\square_{\lambda, < \text{cf}(\lambda)}$ holds, CFM will give a stationary subset of S_U not reflecting at any point of cofinality $\geq \text{cf}(\lambda)$, contradicting Prikry–Silver. $\square$

Theorem (J., Oren, Rinot)

Assuming a supercompact cardinal is consistent, there is a model of $\text{ZFC} + \text{GCH}$ in which for every singular λ :

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2. every family of $< \text{cf}(\lambda)$ stationary subsets of $E_{>\text{cf}(\lambda)}^{\lambda^+}$ reflects simultaneously;
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3. $\square_{\lambda, <\text{cf}(\lambda)}$ fails, while $\square_{\lambda, \text{cf}(\lambda)}$ holds;
4. for every graph $\mathcal{G}$ of size λ^+ and every cardinal $\theta \in E_{>\text{cf}(\lambda)}^{\lambda}$: if every subgraph of $\mathcal{G}$ of size $< \lambda$ has chromatic number $\leq \theta$, then $\text{Chr}(\mathcal{G}) \leq \theta$;
5. for graphs $\mathcal{G}, \mathcal{H}$ of size $\leq \lambda^+$: $\min\{\text{Chr}(\mathcal{G}), \text{Chr}(\mathcal{H})\} \geq \lambda \Rightarrow \text{Chr}(\mathcal{G} \times \mathcal{H}) \geq \lambda$;
6. if $\text{cf}(\lambda) > \omega$: after adding λ Cohen reals, $\mathfrak{u}_{\kappa} < 2^{\kappa}$ for every successor $\kappa < \lambda$ of a singular of cofinality $\neq \text{cf}(\lambda)$.

Definition ($\square_\lambda^B$, Baumgartner's square)

There is $\langle C_\beta \mid \beta < \lambda^+ \text{ limit} \rangle$ with C_β a club in β , $\text{otp}(C_\beta) \leq \lambda$, and if $\text{otp}(C_\beta) = \lambda$, then $C_\beta \cap \alpha = C_\alpha$ for every $\alpha \in \text{acc}(C_\beta)$.

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Ground model V

We start from the model of Apter–Cummings (2000): GCH holds, κ is κ^{++} -supercompact, and $\square_\lambda^B$ holds for every inaccessible $\lambda \leq \kappa$.

Apter–Cummings–Schimmerling (cf. Apter–Cummings (2000), Cummings–Schimmerling (2002))

If $V \subseteq W$, λ is inaccessible in V with $\square_{\lambda}^B$, singular in W , and $(\lambda^+)^V = (\lambda^+)^W$, then $W \models \square_{\lambda, \text{cf}(\lambda)}$.

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All the other clauses (2)–(6) follow from clause (1).

Inamdar–Rinot (2026)

Let D be a ν -complete uniform filter on a regular κ . If some family of fewer than ν stationary subsets of $\{\alpha < \kappa \mid D \text{ is } \text{cf}(\alpha)\text{-indecomposable}\}$ does not reflect simultaneously, then every D -positive set splits into κ many D -positive sets; in particular, D is not an ultrafilter.

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Clause 2

In our model, for every singular λ , every family of fewer than $\text{cf}(\lambda)$ stationary subsets of $E_{>\text{cf}(\lambda)}^{\lambda+}$ reflects simultaneously.

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Clause 2

In our model, for every singular λ , every family of fewer than $\text{cf}(\lambda)$ stationary subsets of $E_{>\text{cf}(\lambda)}^{\lambda^+}$ reflects simultaneously.

Proof.

Let U be the ultrafilter on λ^+ from clause 1. Then $\{\alpha < \lambda^+ \mid U \text{ is } \text{cf}(\alpha)\text{-indecomposable}\} \supseteq E_{>\text{cf}(\lambda)}^{\lambda^+}$, so we apply Inamdar–Rinot with $\nu = \omega$. □

- Recall the ground model V from Apter–Cummings (2000): GCH, κ is κ^{++} -supercompact, and $\square_\lambda^B$ holds for every inaccessible $\lambda \leq \kappa$.
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- Classically, these choices come from a *guiding generic*: an M_U -generic for one fixed ultrapower M_U .
- Here many measures $U_{\kappa,i}$, $i < \kappa^{+3}$, are used at once, so we need a strong kind of guiding generic, which we call a **guru**.

For $x \in \mathcal{P}_\kappa(\kappa^+)$ write $\kappa_x := \text{otp}(\kappa \cap x)$, and $\mathcal{C}_{\kappa,x} := \text{Col}((\kappa_x)^{++}, < \kappa)$.

Definition (Guru for κ)

A sequence $\vec{t} = \langle t_i \mid i < \kappa^{++} \rangle$ such that:

1. $t_i(x) \in \mathcal{C}_{\kappa,x}$ for every relevant $x \in \mathcal{P}_\kappa(\kappa^+)$;
2. for $j < i$, $\{x \mid t_j(x) \subseteq t_i(x)\}$ contains a club in $\mathcal{P}_\kappa(\kappa^+)$;
3. for every $x \mapsto D(x)$ with $D(x)$ dense in $\mathcal{C}_{\kappa,x}$, there is an i such that $t_i(x) \in D(x)$ for all x .

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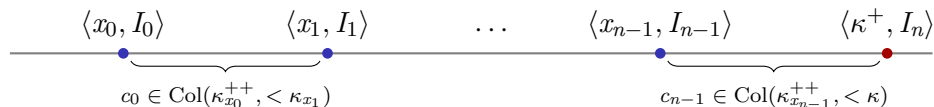
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- For **every** supercompact measure U on $\mathcal{P}_\kappa(\kappa^+)$, $\{[t_i]_U \mid i < \kappa^{++}\}$ generates an $(M_U, \text{Col}(\kappa^{++}, < j_U(\kappa)))$ -generic filter: a guiding generic for all measures at once.
- Existence: if $2^{\kappa^+} = \kappa^{++}$, diagonalise along an enumeration of all D 's, using diagonal intersections of clubs at limits.

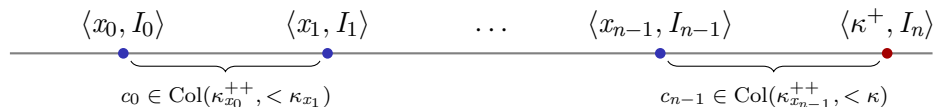
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$x_0 \subsetneq x_1 \subsetneq \dots$ in $\mathcal{P}_\kappa(\kappa^+)$ with $\kappa_x := \text{otp}(\kappa \cap x)$; each I_i is an *implicit guru*: $\text{dom}(I_i)$ is in every measure at x_i , and $I_i \subseteq t_{x_i,j}$ for some j .

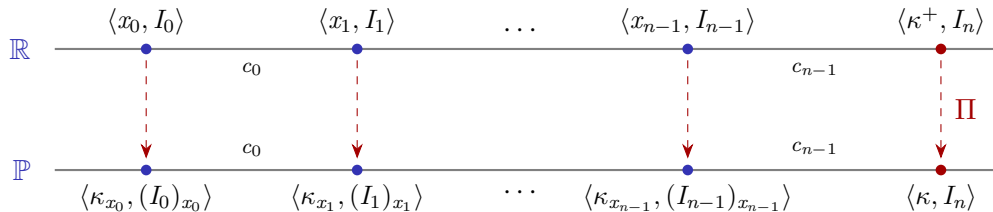
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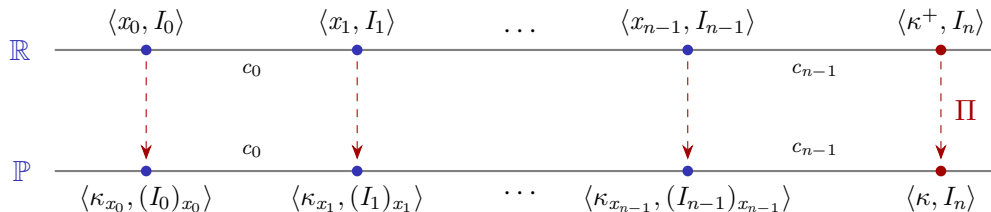


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The role of gurus: compatibility

Two implicit gurus $I \subseteq t_l$, $I' \subseteq t_{l'}$ are both extended by $t_{\max(l,l')}$ on a club, hence on a set in every measure. So conditions with the same stem and compatible collapses are compatible. Hence $\mathbb{R}$ is κ^{++} -Knaster.



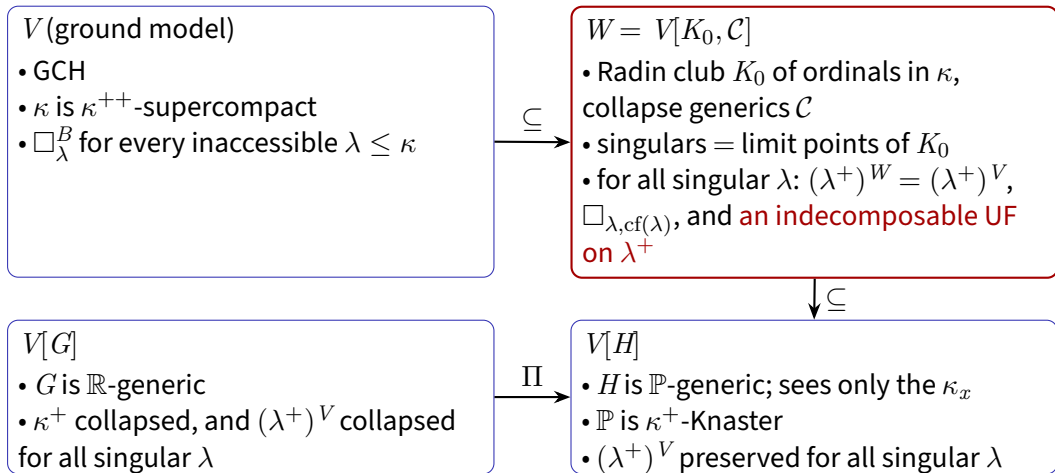


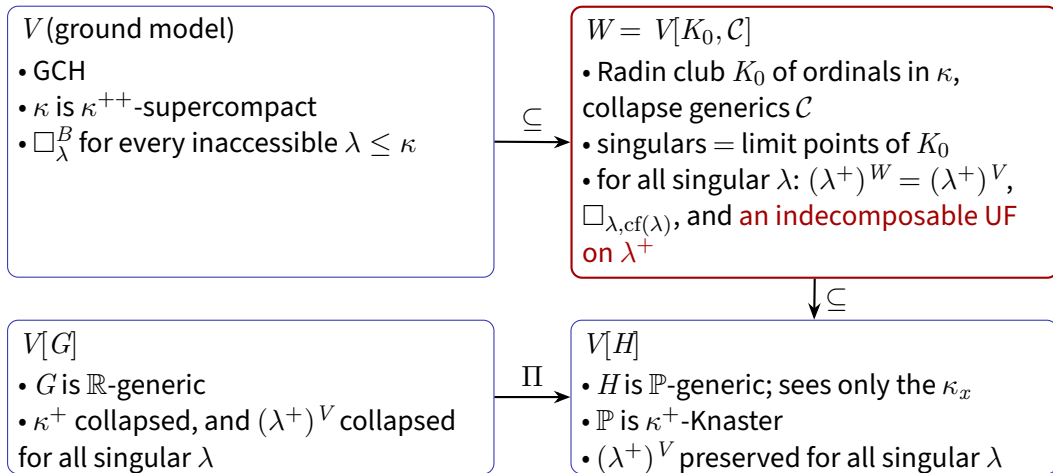
- $\mathbb{P}$ has the same shape as $\mathbb{R}$ and the same collapses, but remembers only the ordinals κ_x .
- $I_x(y) := I(\pi_x[y])$, where $\pi_x : \text{otp}(x) \rightarrow x$ is the inverse collapsing map of x : the same information as I , with the domain reflected down from $\mathcal{P}_{\kappa_x}(x)$ to $\mathcal{P}_{\kappa_x}(\kappa_x^+)$.
- $\Pi : \mathbb{R} \rightarrow \mathbb{P}$ is a projection. Let G be $\mathbb{R}$ -generic and $H := \Pi[G]$.

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- κ -Linked₀: there are only κ many possible stems $\langle c_{-1}, \kappa_0, c_0, \dots \rangle$, and conditions with the same stem are compatible, since implicit gurus always amalgamate. So $\mathbb{P}$ is κ^+ -Knaster; likewise its part below λ is λ^+ -cc.

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- Hence $(\lambda^+)^V$ is preserved in $V[H]$ for every singular λ , while in $V[G]$ it is collapsed, since the stems of conditions in G cover $(\lambda^+)^V$ by at most λ sets of size $< \lambda$.





G is $\mathbb{R}$ -generic over V , and $H := \Pi[G]$ is $\mathbb{P}$ -generic over V . K_0 and $\mathcal{C}$ are read off from H , so $W \subseteq V[H] \subseteq V[G]$. The final model is $W' := V_\kappa^W$.

- $\text{Aut}_\kappa(\kappa^+)$: bijections of κ^+ fixing $[0, \kappa]$ and a tail. Each $\Gamma \in \text{Aut}_\kappa(\kappa^+)$ acts on $\mathbb{R}$ by moving the x 's and the measure-one sets. Since Γ fixes measure-one many points, it maps implicit gurus to implicit gurus.

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Proposition (weak homogeneity)

If $p, q \in \mathbb{R}$ have the same ordinals κ_{x_i} and pairwise compatible collapses, then $\Gamma(p) \parallel q$ for some $\Gamma \in \text{Aut}_\kappa(\kappa^+)$.

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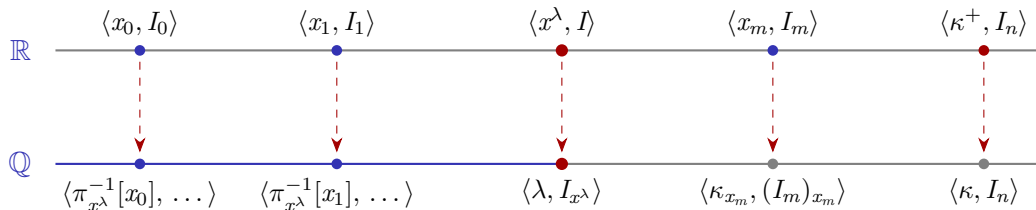
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- Γ fixes all ordinals below κ , hence the canonical names of the Radin club K_0 and of the collapse generics $\mathcal{C}$. Let $W := V[K_0, \mathcal{C}]$, so $W \subseteq V[H] \subseteq V[G]$.
- Every set in W has a Γ -invariant name, so whether it has a property is decided by the K_0 - and $\mathcal{C}$ -part of a condition alone.

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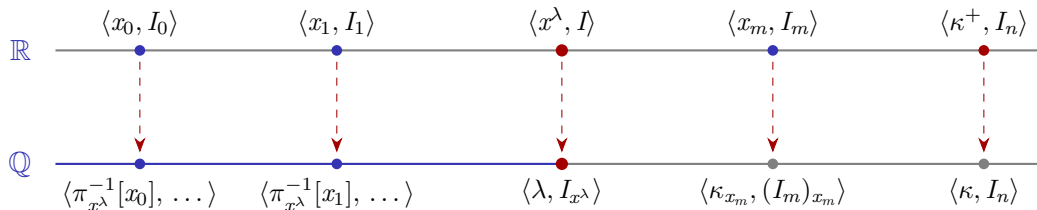
From $\mathbb{R}$ to $\mathbb{Q} := \mathbb{Q}_{0, x^\lambda}$



Below x^λ , $\mathbb{Q}$ keeps the sets of $\mathbb{R}$, transported by the transitive collapse $\pi_{x^\lambda}^{-1}$ to subsets of $(\lambda^+)^V$; above λ it keeps only the *ordinals*, as in $\mathbb{P}$.

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The projections $\mathbb{R} \rightarrow \mathbb{Q} \rightarrow \mathbb{P}$ compose to Π . Let G^λ be the $\mathbb{Q}$ -generic induced by G ; then $W \subseteq V[H] \subseteq V[G^\lambda] \subseteq V[G]$.

In $V[G^\lambda]$

The sets in the $\mathbb{R}$ -parts of conditions in G^λ form a $\subseteq$ -increasing chain $\langle x_\nu \mid \nu < \delta \rangle$ in $\mathcal{P}_\lambda(\lambda^+)$ with union λ^+ , and $\text{cf}(\delta) = \text{cf}(\lambda)$. Fix in W a cofinal sequence $\langle \nu_\tau \mid \tau < \text{cf}(\lambda) \rangle$ in δ , put $y_\tau := x_{\nu_\tau}$, and let $F_\lambda := \{A \subseteq \mathcal{P}_\lambda(\lambda^+) \mid \exists \tau_0 < \text{cf}(\lambda) \forall \tau \in [\tau_0, \text{cf}(\lambda)) (y_\tau \in A)\}$.

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Lemma

$F_\lambda \cap W \in W$.

Idea. Each $A \in W$ has a name invariant under automorphisms fixing $K_0, \mathcal{C}$; such automorphisms make any two conditions with the same K_0 -, $\mathcal{C}$ -information compatible, so “ $A \in F_\lambda$ ” is decided by that information.

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In W

Extend $\{\sup[A] \mid A \in F_\lambda \cap W\}$ to an ultrafilter $\mathcal{W}_\lambda$ on λ^+ ; it is uniform, as each $\sup[A]$ contains a tail of the cofinal sequence $\langle \sup(y_\tau) \mid \tau < \text{cf}(\lambda) \rangle$.

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Case 1: θ is regular in $V[G^\lambda]$

- In $V[G^\lambda]$, $\tau \mapsto f(\sup y_\tau)$ takes only $\text{cf}(\lambda) < \theta$ values, so all of them lie below some $\alpha < \theta$.
- $A := \{y \in \mathcal{P}_\lambda(\lambda^+) \mid f(\sup y) < \alpha\}$ is in W (it is defined from f and α) and contains every y_τ . So $A \in F_\lambda \cap W$.
- Hence $X := f^{-1}[\alpha] \supseteq \sup[A]$ is in $\mathcal{W}_\lambda$, and $|f[X]| \leq |\alpha| < \theta$.

Let $\theta \in (\text{cf}(\lambda), \lambda)$ and $f: \lambda^+ \rightarrow \theta$ in W . We find $X \in \mathcal{W}_\lambda$ with $|f[X]| < \theta$.

Case 1: θ is regular in $V[G^\lambda]$

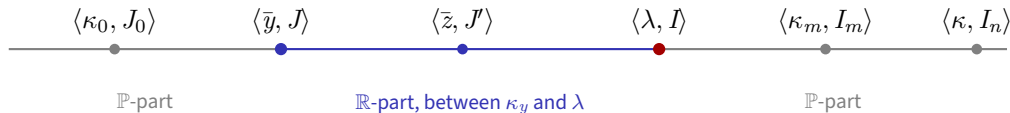
- In $V[G^\lambda]$, $\tau \mapsto f(\sup y_\tau)$ takes only $\text{cf}(\lambda) < \theta$ values, so all of them lie below some $\alpha < \theta$.
- $A := \{y \in \mathcal{P}_\lambda(\lambda^+) \mid f(\sup y) < \alpha\}$ is in W (it is defined from f and α) and contains every y_τ . So $A \in F_\lambda \cap W$.
- Hence $X := f^{-1}[\alpha] \supseteq \sup[A]$ is in $\mathcal{W}_\lambda$, and $|f[X]| \leq |\alpha| < \theta$.

This fails if θ is regular in W but collapsed in $V[G^\lambda]$, e.g. $\theta = (\mu^+)^V$ for a singular $\mu < \lambda$ of W .

Case 2: θ is regular in W but not in $V[G^\lambda]$

20 / 33

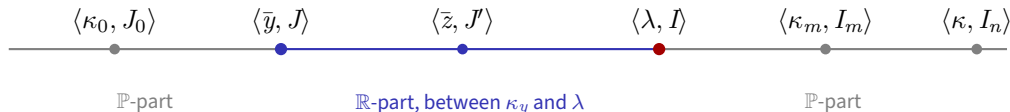
Pick a point $y \subseteq x^\lambda$ of the chain with $\theta < \kappa_y$, and let $\bar{y} := \pi_{x^\lambda}^{-1}[y]$. Use $\mathbb{Q}_{y,x^\lambda}$:



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20 / 33

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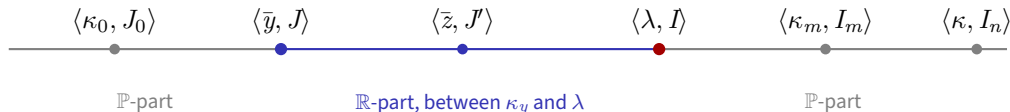


Let G^{y,x^λ} be the $\mathbb{Q}_{y,x^\lambda}$ -generic induced by G . Below κ_y it acts like $\mathbb{P}$, so θ is a regular cardinal in $V[G^{y,x^\lambda}]$.

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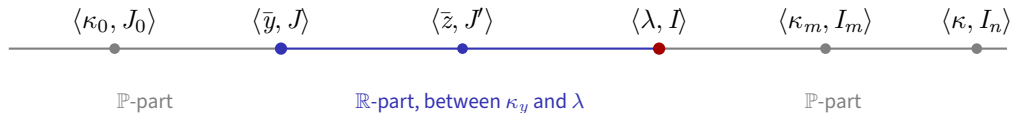
A new filter contained in $\mathcal{W}_\lambda$

The points of the chain above $\bar{y}$ are in $V[G^{y, x^\lambda}]$, and so is the tail $\langle y_\tau \mid \tau \geq \tau_y \rangle$ (the indices ν_τ are in W). Its tail filter F_λ^y has the same tails, so $F_\lambda^y \cap W = F_\lambda \cap W$, and $\{\sup[A] \mid A \in F_\lambda^y \cap W\} \subseteq \mathcal{W}_\lambda$. Now run Case 1 in $V[G^{y, x^\lambda}]$ with F_λ^y .

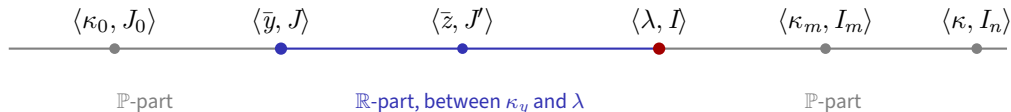
Case 3: θ is singular in W

21 / 33

Again pick $y \subseteq x^\lambda$ with $\theta < \kappa_y$, and work in $V[G^{y, x^\lambda}]$:



Again pick $y \subseteq x^\lambda$ with $\theta < \kappa_y$, and work in $V[G^{y, x^\lambda}]$:



1. Factor $\mathbb{Q}_{y, x^\lambda}$ at κ_y . Since $\theta < \kappa_y$, the quotient $\mathbb{Q}_{y, x^\lambda} / W$ is $\leq^*$ - θ^+ -closed and has the Prikry property, so it adds no new subsets of θ .
2. Let $a := \{f(\sup y_\tau) \mid \tau_y \leq \tau < \text{cf}(\lambda)\} \subseteq \theta$. Then $|a| \leq \text{cf}(\lambda) < \theta$, and $a \in W$ by 1.
3. So $A := \{y' \in \mathcal{P}_\lambda(\lambda^+) \mid f(\sup y') \in a\} \in W$, and it contains y_τ for all $\tau \geq \tau_y$. Thus $A \in F_\lambda^y \cap W = F_\lambda \cap W$.
4. Hence $X := f^{-1}[a] \supseteq \sup[A]$ is in $\mathcal{W}_\lambda$, and $|f[X]| \leq |a| < \theta$. □

Definition

A *graph* is $\mathcal{G} = (G, E)$ with E an irreflexive symmetric relation on G . A colouring $c : G \rightarrow \mu$ is *good* iff $c(x) \neq c(y)$ whenever $x E y$. The *chromatic number* $\text{Chr}(\mathcal{G})$ is the least μ admitting a good colouring $c : G \rightarrow \mu$.

Definition (Keisler)

A uniform ultrafilter U on κ is (λ, κ) -regular iff there are $A_\beta \in U$ ($\beta < \kappa$) such that every $\alpha < \kappa$ lies in fewer than λ of them, i.e. $\bigcap_{\beta \in B} A_\beta = \emptyset$ for every $B \in [\kappa]^\lambda$.

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So the regular sets A_β come for free; what we add is indecomposability.

Clause 4

In our model, let λ be singular, $\mathcal{G}$ a graph of size λ^+ , and $\theta < \lambda$ a cardinal with $\text{cf}(\theta) > \text{cf}(\lambda)$. If every subgraph of $\mathcal{G}$ of size less than λ has chromatic number at most θ , then $\text{Chr}(\mathcal{G}) \leq \theta$.

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Let $\mathcal{G} = (\lambda^+, E)$ and U the ultrafilter from clause 1.

Step 1: local colourings

By Kanamori, fix $A_\beta \in U$ ($\beta < \lambda^+$) such that every α lies in fewer than λ of them. So $G_\alpha := \{\beta \mid \alpha \in A_\beta\}$ has size $< \lambda$; fix a good colouring $c_\alpha : G_\alpha \rightarrow \theta$.

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Step 2: colour by ultrafilter classes

Let $f_\beta : A_\beta \rightarrow \theta$ be $f_\beta(\alpha) := c_\alpha(\beta)$, and colour $\beta \mapsto [f_\beta]_U$. If $\beta E \beta'$, then for every $\alpha \in A_\beta \cap A_{\beta'} \in U$ we have $c_\alpha(\beta) \neq c_\alpha(\beta')$, so $[f_\beta]_U \neq [f_{\beta'}]_U$: the colouring is good.

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So $\text{Chr}(\mathcal{G}) \leq$ the number of U -classes of functions $\lambda^+ \rightarrow \theta$; it remains to count them.

Silver (1974)

Let $\chi = \text{cf}(\chi) < 2^\chi < \nu \leq \kappa$, and U a χ -incomplete, $[\chi, \nu)$ -indecomposable uniform ultrafilter over κ . Then there are $\vartheta < \chi$ and $\phi : \kappa \rightarrow \vartheta$ such that for every $f : \kappa \rightarrow \mu$ with $\mu < \nu$ there is $g : \vartheta \rightarrow \mu$ with $f = g \circ \phi \pmod{U}$.

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Step 3: count

Take $\kappa := \lambda^+$, $\chi := \text{cf}(\lambda)^+$ and $\nu := \lambda$; by GCH $2^\chi < \lambda$, and U is $[\chi, \lambda)$ -indecomposable by clause 1. (If U is χ -complete, then since U is $[\chi, \theta]$ -indecomposable, U is θ^+ -complete. Then every $f : \lambda^+ \rightarrow \theta$ is constant mod U , and there are $\leq \theta$ classes.) Otherwise every f_β is $g_\beta \circ \phi \pmod{U}$ with $g_\beta : \vartheta \rightarrow \theta$ and $\vartheta \leq \text{cf}(\lambda)$, so there are at most $\theta^{\text{cf}(\lambda)} = \theta$ classes, by GCH and $\text{cf}(\theta) > \text{cf}(\lambda)$. □

Definition

The product $\mathcal{G} \times \mathcal{H}$ has vertices $V(\mathcal{G}) \times V(\mathcal{H})$, with (g, h) adjacent to (g', h') iff $g E_{\mathcal{G}} g'$ and $h E_{\mathcal{H}} h'$.

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- Always $\text{Chr}(\mathcal{G} \times \mathcal{H}) \leq \min\{\text{Chr}(\mathcal{G}), \text{Chr}(\mathcal{H})\}$. Hedetniemi (1966) conjectured equality for finite graphs; Shitov disproved this in 2019.
- **Hajnal (1985)**. For every λ there are $\mathcal{G}, \mathcal{H}$ with $\text{Chr} = \lambda^+$ but $\text{Chr}(\mathcal{G} \times \mathcal{H}) = \lambda$: a gap of one cardinal.
- **Soukup (1988)**. A gap of 2 is consistent.
- **Rinot (2017)**. In L , one has arbitrarily large gaps.

Clause 5

In our model, for every singular λ and graphs $\mathcal{G}, \mathcal{H}$ of size at most λ^+ : if $\min\{\text{Chr}(\mathcal{G}), \text{Chr}(\mathcal{H})\} \geq \lambda$, then $\text{Chr}(\mathcal{G} \times \mathcal{H}) \geq \lambda$.

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For infinitely chromatic $\mathcal{G}, \mathcal{H}$: every subgraph of $\mathcal{G}$ of size less than $\text{Chr}(\mathcal{H})$ has chromatic number $\leq \text{Chr}(\mathcal{G} \times \mathcal{H})$.

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Proof of clause 5.

Suppose $\lambda \leq \text{Chr}(\mathcal{G}_1) \leq \text{Chr}(\mathcal{G}_0)$, with $\mathcal{G}_0, \mathcal{G}_1$ of size λ^+ , but $\mu := \text{Chr}(\mathcal{G}_0 \times \mathcal{G}_1) < \lambda$.

- By Hajnal's lemma, every subgraph of $\mathcal{G}_0$ of size $< \lambda$ has chromatic number $\leq \mu$.
- By the proof of clause 4 and as λ is a strong limit: $\lambda \leq \text{Chr}(\mathcal{G}_0) \leq \mu^{\text{cf}(\lambda)} < \lambda$, a contradiction.



Hajnal (1985), in ZFC

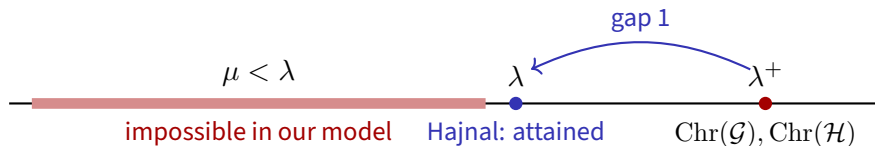
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For every singular λ : if $\mathcal{G}, \mathcal{H}$ are graphs on λ^+ with $\text{Chr}(\mathcal{G}) = \text{Chr}(\mathcal{H}) = \lambda^+$, then $\text{Chr}(\mathcal{G} \times \mathcal{H}) \geq \lambda$. This is clause 5.

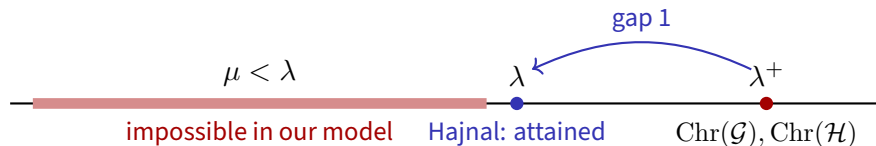


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Hajnal's gap 1 is best possible in ZFC.

Definition

$\mathfrak{u}_\kappa$ is the least size of a base of a uniform ultrafilter on κ . Always $\mathfrak{u}_\kappa \leq 2^\kappa$.

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The key input is a theorem of Raghavan–Shelah on indecomposable ultrafilters and Cohen reals.

Raghavan–Shelah (2020)

Let $\aleph_0 < \text{cf}(\mu) < \kappa < \mu$, μ a singular strong limit, U a $\text{cf}(\mu)$ -indecomposable uniform UF on κ . After adding μ Cohen reals, every uniform UF on κ extending U has a base of size $\leq \mu$.

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Lemma

Let μ be a singular strong limit with $\theta := \text{cf}(\mu) > \omega$, and $\sigma < \mu$ singular with a θ -indecomposable uniform UF on σ^+ . After adding μ Cohen reals, $\mathfrak{u}_{\sigma^+} < 2^{\sigma^+}$.

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Proof.

Apply Raghavan–Shelah with $\kappa := \sigma^+$: $\mathfrak{u}_{\sigma^+} \leq \mu$. On the other hand $2^{\sigma^+} \geq 2^{\aleph_0} \geq \mu$, and $\text{cf}(2^{\sigma^+}) > \sigma^+ > \theta = \text{cf}(\mu)$, so $2^{\sigma^+} \neq \mu$. Hence $2^{\sigma^+} > \mu \geq \mathfrak{u}_{\sigma^+}$. $\square$

Clause 6

In our model, let λ be singular with $\text{cf}(\lambda) > \omega$. After adding λ Cohen reals, $\mathfrak{u}_{\sigma^+} < 2^{\sigma^+}$ for every singular $\sigma < \lambda$ with $\text{cf}(\sigma) < \text{cf}(\lambda) < \sigma$.

Clause 6

In our model, let λ be singular with $\text{cf}(\lambda) > \omega$. After adding λ Cohen reals, $\mathfrak{u}_{\sigma^+} < 2^{\sigma^+}$ for every singular $\sigma < \lambda$ with $\text{cf}(\sigma) < \text{cf}(\lambda) < \sigma$.

Proof of clause 6

Let λ be singular with $\text{cf}(\lambda) > \omega$. In our model λ is a strong limit (GCH), so take $\mu := \lambda$ and $\theta := \text{cf}(\lambda)$ in the lemma.

For every singular $\sigma < \lambda$ with $\text{cf}(\sigma) < \text{cf}(\lambda) < \sigma$, the ultrafilter on σ^+ from clause 1 is $\text{cf}(\lambda)$ -indecomposable, since $\text{cf}(\lambda) \in (\text{cf}(\sigma), \sigma)$.

Hence, after adding λ Cohen reals, $\mathfrak{u}_{\sigma^+} < 2^{\sigma^+}$ for all such σ , simultaneously.

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Thank you

and happy 80th birthday, Menachem!